Technical Notes

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Analysis of Model Equations of Gas Dynamics

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Introduction

THIS paper explores application of the decomposition method[1] as a new approach to analysis of equations in gas dynamics with given initial data. This methodology is of interest since it has solved a wide variety of nonlinear equations without resorting to linearizing techniques; it avoids discretization, and, in a number of problems, offers accurate solutions to the actual nonlinear equation. The avoidance of discretization, linearization, perturbation, and its generality offer advantages not otherwise available. It has been applied to linear or nonlinear and/or stochastic differential or partial differential equations, or systems of such equations even when linear, nonlinear, or coupled boundary conditions are involved.

We will consider a previously studied model equation:[2]

$$u_t + \tfrac{1}{2}(u^2)_x = u(1-u)$$

For the initial conditions, we use $u(x,0) = g(x) = b(1-e^{-x})$ where $b > 0$ as chosen in Ref. 2. The solution depends on the condition and is carried out in the same manner for other conditions. Let

$$L = \frac{\partial}{\partial t}$$

and write

$$Lu = u - u^2 - \tfrac{1}{2}(u^2)_x$$

Defining

$$L^{-1} = \int_0^t [\cdot]\, dt$$

and operating on both sides with L^{-1}, we obtain

$$u = u(x,0) + L^{-1}u - L^{-1}u^2 - \tfrac{1}{2}L^{-1}(u^2)_x$$

Using the decomposition, let

$$u = \sum_{n=0}^{\infty} u_n$$

and expand u^2 and $(u^2)_x$ in the A_n polynomials defined in Ref. 1 and elsewhere. Briefly, these are generated for the specific nonlinearity so that if we have an analytic function or composite of such functions, a convergent series of polynomials, A_n, can be developed such that the A_n for any n depends only on $u_0, u_1, \ldots, u_n$, rather than all the components of u. For u^2 we have[1,2]

$$A_0 = u_0^2$$
$$A_1 = 2u_0u_1$$
$$A_2 = u_1^2 + 2u_0u_2$$
$$\vdots$$

and for $(u^2)_x$ we have

$$A_0 = u_{0_x}^2$$
$$A_1 = 2u_{0_x}u_{1_x}$$
$$A_2 = u_{1_x}^2 + 2u_{0_x}u_{2_x}$$
$$\vdots$$

Therefore[1]

$$u_0 = g(x) = b(1-e^{-x})$$
$$u_1 = L^{-1}u_0 - L^{-1}u_0^2 - \tfrac{1}{2}L^{-1}u_{0_x}^2$$
$$u_2 = L^{-1}u_1 - L^{-1}(2u_0u_1) - \tfrac{1}{2}L^{-1}(2u_{0_x}u_{1_x})$$
$$u_3 = L^{-1}u_2 - L^{-1}(u_1^2 + 2u_0u_2) - \tfrac{1}{2}L^{-1}(u_{1x}^2 + 2u_{0_x}u_{2_x})$$
$$\vdots$$

Consequently, we have

$$u_0 = b(1-e^{-x})$$
$$u_1 = bt(1-e^{-x}) - b^2t(1-e^{-x})^2 - \tfrac{1}{2}b^2te^{-2x}$$
$$u_2 = \frac{bt^2}{2}(1-e^{-x}) - \frac{b^2t^2}{2}(1-e^{-x})^2 - \frac{b^2t^2}{4}e^{-2x} - b^2t^2(1-e^{-x})^2 - b^3t^2(1-e^{-x})^3 - \frac{b^3t^2}{2}e^{-2x}(1-e^{-2x})$$
$$\vdots$$

Thus, for the given initial conditions:

$$u = b(1-e^{-x}) + bt(1-e^{-x}) - b^2t(1-e^{-x})^2 - \tfrac{1}{2}b^2te^{-2x} + \frac{bt^2}{2}(1-e^{-x}) + \frac{b^2t^2}{2}(1-e^{-x})^2 - \frac{b^2t^2}{4}e^{-2x} - b^2t^2(1-e^{-x})^2 - b^3t^2(1-e^{-x})^3 - \frac{b^3t^2}{2}e^{-2x}(1-e^{-2x}) + \cdots$$

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Table 1 Solution by decomposition for N-term approximant ϕ_n

x	ϕ_3	ϕ_6	ϕ_9	ϕ_{12}	$D\phi_{12}$
−1.0	0.0	0.0	0.0	0.0	2.000000
−0.8	0.258364	0.253792	0.254207	0.254206	2.000000
−0.6	0.281425	0.296847	0.296396	0.296396	2.000000
−0.4	0.089598	0.147337	0.146348	0.146349	2.000000
−0.2	−0.096523	−0.024947	−0.025887	−0.025886	2.000000
0	−0.182144	−0.118808	−0.119393	−0.119393	2.000000
0.2	−0.189972	−0.135439	−0.135649	−0.135649	2.000000
0.4	−0.177398	−0.114085	−0.113970	−0.113969	2.000000
0.6	−0.191415	−0.083857	−0.083321	−0.083321	2.000000
0.8	−0.226097	−0.524057	−0.050944	−0.050944	2.000000
1.0	0.0	0.0	0.0	0.0	2.000000

We note that we can choose a different initial condition and proceed in the same manner; the result obtained depends upon the initial condition to yield a solution in the form of a convergent series in some region. Other forms of gas dynamic model equations are solvable in the same manner. Different nonlinear terms simply result in different A_n polynomials. Forcing terms and initial/boundary conditions modify the u_0 term and the resulting series.

Possible modifications to the model equations are numerous; the many possibilities will not be dealt with here. General procedures have been fully discussed in Ref. 1 and should be applicable with possible decreased demands on computer resources. The primary problem is realistic modeling rather than tractability of equations.

To show possible accuracy, let us consider a simple one-dimensional example:

$$\frac{d^2u}{dx^2} - 40xu = 2$$

$$u(-1) = u(1) = 0$$

This equation is a generalization of Airy's equation to a nonhomogeneous case with the added difficulty that the large numerical coefficient of xu makes the equation relatively stiff. The nonzero forcing function yields an additional Airy-like function. Let

$$L = \frac{d^2}{dx^2}$$

and write

$$Lx = 2 + 40xu$$

$$u = c_1 + c_2x + L^{-1}(2) + L^{-1}(40xu)$$

Let

$$u_0 = c_1 + c_2x + L^{-1}(2) = c_1 + c_2x + x^2 \quad \text{and} \quad u = \sum_{n=0}^{\infty} u_n$$

The following components of u are given by

$$u_{n+1} = L^{-1}40xu_n$$

for $n \geq 0$. Thus,

$$u_1 = L^{-1}40xu_0 = \frac{20}{3}c_1x^3 + \frac{10}{3}c_2x^4 + 2x^5$$

Proceeding in the same way:

$$u_2 = \frac{80}{9}c_1x^6 + \frac{200}{63}c_2x^7 + \frac{10}{7}x^8$$

We can continue in the same manner to determine the u_i as far as we like. The sum

$$u = \sum_{n=0}^{\infty} u_n$$

is the true solution and some n-term approximation,

$$\phi_n = \sum_{i=0}^{n-1} u_i$$

will be sufficient—usually for low n. The constants are evaluated by using ϕ_n for u in satisfying the specified conditions.

So far we have the three-term approximation, ϕ_3, given by:

$$\phi_3 = c_1 + c_2x + x^2 + \frac{20}{3}c_1x^3 + \frac{10}{3}c_2x^4 + 2x^5 + \frac{80}{9}c_1x^6 + \frac{200}{63}c_2x^7 + \frac{10}{7}x^8$$

Imposing the boundary conditions at this stage:

$$\phi_3(1) = c_1\left(1 + \frac{20}{3} + \frac{80}{9}\right) + c_2\left(1 + \frac{10}{3} + \frac{200}{63}\right) + \left(1 + 2 + \frac{10}{7}\right) = 0$$

$$\phi_3(-1) = c_1\left(1 - \frac{20}{3} + \frac{80}{9}\right) + c_2\left(-1 + \frac{10}{3} - \frac{200}{63}\right) + \left(1 - 2 + \frac{10}{7}\right) = 0$$

We find

$$c_1 = -\frac{13779}{75649} \quad \text{and} \quad c_2 = -\frac{2034}{10807}$$

The limit of ϕ_n as $n \to \infty$ is obviously u. Table 1 shows values of the n-term approximant ϕ_n by the decomposition method for $n = 12$. Computation of

$$Du = \left(\frac{d^2}{dx^2} - 40x\right)u$$

should equal 2.0 if our approximation is a good one. (Calculation of more terms of ϕ_n will make the results even more striking and can easily be carried further.)

Thus, if we compute analytic derivatives, we have $D\phi_{12} = 2.000000$ to seven-digit accuracy. (The only reason for going as far as ϕ_{12} is that the number 40 in the coefficient makes the system stiff.) Our conclusion is that the decomposition is not only easy to use but extremely accurate. The referenced work shows that it also is easily extended to nonlinear multidimensional cases difficult and in some cases impossible by other methods.

An additional advantage is that the procedure works as well in the multidimensional case, including nonlinear terms, which are then written as sums of the appropriate Adomian polynomials as discussed in Ref. 1. The procedure is also easily extended to $\nabla^2 u = f(x,y,z) + ku$ or even $\nabla^2 u + Nu = f(x,y,z) + ku$ where Nu is an analytic term.

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References

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[2]Salas, M. D., Abarbanel, S., and Gottlieb, D., "Multiple Steady States for Characteristic Initial-Value Problems," Institute of Computer Applications in Science and Engineering Report, Langley Research Center, Virginia, 1986.

Numerical Evaluation of Whitham's F-Function for Supersonic Projectiles

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Introduction

In 1952 Whitham[1] presented a nonlinear analysis accurate to first-order for determination of the complete supersonic flowfield around slender and smooth bodies of revolution, and this analysis has been used extensively to obtain flowfields about supersonic projectiles, missiles, and aircraft. At the center of this analysis is his F-function

$$F(y) = \frac{1}{2\pi}\int_0^y \frac{A''(x)}{\sqrt{y-x}}\,dx \tag{1}$$

where $A(x)$ defines the cross-sectional area of the body with axial distance x measured from the body nose, and $A''(x)$ is its double derivative. Whitham's variable y defines the origin of a characteristic curve emanating from the body surface, by being equal to $x - \beta r$ at the surface, where $\beta = (M_\infty^2 - 1)^{1/2}$. Physically, this is the axial distance from the body nose to a point at which a characteristic curve, when extended backward from the surface, intersects the x axis. For more complex bodies such as aircraft, $A(x)$ may be considered as an effective cross-sectional area that includes lift and other effects.

In using Whitham's analysis, it is obvious that much of the work is centered on determining $F(y)$ by integration. If the cross-sectional area of the body is an extremely simple function of distance, then $F(y)$ may be obtained by analytic integration. The shapes of real projectiles, missiles, and aircraft, however, are normally too complicated for analytic integrations, and one must resort to a numerical integration procedure.

Carlson[2] developed a numerical integration procedure that has been used since the 1960s. In this method the body is subdivided into many segments of equal or unequal lengths. The area variations of these segments are modeled as a series of piecewise, continuous, parabolic arcs. In this case, $A''(x)$ is a constant for each segment, and the piecewise analytic integrations to obtain $F(y)$ can be done easily, constituting the major advantage of this method.

A variation of Carlson's scheme was introduced by Igoe,[3] primarily to improve accuracy of the numerical integration. Igoe's procedure uses more accurate second derivatives than Carlson's method. They are obtained from central-difference formulas of finite-difference techniques, which are based on early mathematical work of Richardson.[4] Although this method is more accurate than Carlson's, it is more difficult to use and also requires that the body be subdivided into numerous short segments of equal size. Furthermore, Richardson's extrapolation technique must be used with caution to avoid numerical difficulties for very short segments.

This paper presents an accurate and efficient numerical method for evaluating $F(y)$. It is as accurate as Igoe's method, presents no computational difficulties, and reveals physical insight on how particular body shape features contribute to the $F(y)$. It is more difficult, however, to program in computers than either Carlson's or Igoe's methods.

Numerical Evaluation of $F(y)$

The body is subdivided into a series of segments, normally of unequal lengths, so that any simple-shaped section of arbitrary length can be included efficiently as a single segment. Furthermore, a segment may arbitrarily be made short to closely model a particular, complex part of the body shape, without forcing a subdivision of other segments, as illustrated in Fig. 1a. These modeling features can markedly reduce the total number of body segments compared to the methods of Carlson[2] and Igoe.[3] In the present work the ith segment of a body has a radius given by

$$R_i(x) = \sum_{j=0}^{n_i} a_{i,j}[x - x_i]^j$$

which is a polynomial curve with a finite number of terms (n_i normally less than 5). The sequence of polynomial segments representing the entire body shape is usually piecewise continuous in position and slope at the junctions between segments. For the ith segment this defines the coefficients $a_{i,j}$ for $0 \le j \le n_i$, and for convenience the coefficients $a_{i,j}$ for $n_i < j \le 2n_i$ are set equal to zero.

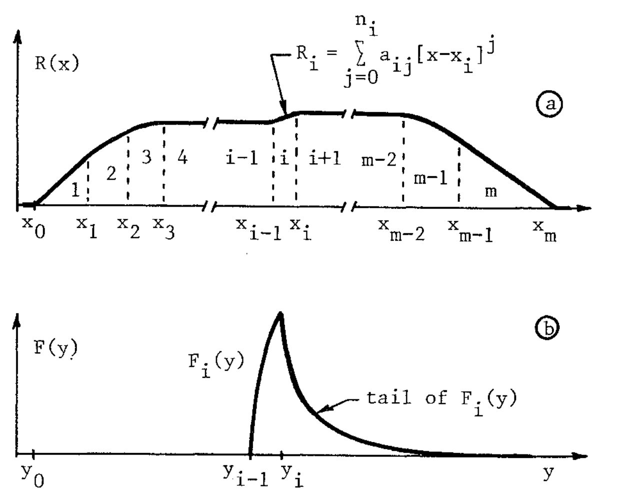

Fig. 1 Body shape defined by m piecewise polynomial segments (a) and the contribution of the ith body segment to the F-function (b).

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